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Inefficiency of Games with Social Context

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Abstract The study of other-regarding player behavior such as altruism and spite in games has recently received quite some attention in the algorithmic game theory literature. Already for very simple models, it has been shown that altruistic behavior can actually be harmful for society in the sense that the price of anarchy may *increase* as the players become more altruistic. In this paper, we study the severity of this phenomenon for more realistic settings in which there is a complex underlying social structure, causing the players to direct their altruistic and spiteful behavior in a refined player-specific sense (depending, for example, on friendships that exist among the players). Our findings show that the increase in the price of anarchy is modest for congestion games and minsum scheduling games, whereas it is drastic for generalized second price auctions.

1 Introduction

Many practical situations involve a group of strategic decision makers who attempt to achieve their own self-interested goals. It is well known that strategic decision making may result in outcomes that are suboptimal for the society as

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a whole. The need to gain an accurate understanding of the extent of suboptimality caused by selfish behavior has led to the study of the *inefficiency of equilibria* in algorithmic game theory. In this context, a common inefficiency measure is the *price of anarchy* [21], which relates the worst-case cost of a Nash equilibrium to the one of an optimal outcome.

More recently, quite some attention has been given to more general settings in which the players do not necessarily behave entirely selfishly, but may alternatively exhibit *spiteful* or *altruistic* behavior; see, for instance, [2,4,5, 7–9,13,16–18]. Studying such alternative behaviors in games is motivated by the observation that altruism and spite are phenomena that frequently occur in real life (see, for example, [14]). Consequently, it is desirable to incorporate such alternative behavior in game-theoretical analyses.

Previous work on the price of anarchy for spiteful and altruistic games has focused on simple models of spite and altruism, where a spite/altruism level α_i is associated to each player i denoting the extent to which his perceived cost is influenced by any nonspecific other player. Already for these simple models it has been observed in a series of papers [5,7,8] that altruistic behavior can actually be harmful in the sense that the price of anarchy may *increase* as players become more altruistic. This observation served as a starting point for the investigations conducted in this paper. The main question that we address here is: How severe can this effect be if one considers more refined models of altruism that capture complex social relationships between the players?

Our Contributions. In this paper, we study a more general player-specific model of spite and altruism. Our model can be viewed as extending a given strategic game by imposing a *social-network structure* on top of the players, which specifies for each pair of players (i, j) an altruism/spite level α_{ij} signifying how much player i cares about player j; these relations are not necessarily symmetric. This allows us to model more realistic settings in which the behavior of the players depends on a complex underlying social structure, expressing friendships and animosities among the players. Our altruistic games fall into the framework of *social context games* proposed in [1].

For this general model of games with altruism and spite, we are interested in studying the price of anarchy. The *smoothness framework*, originally introduced by Roughgarden [23], has become a standard method for proving upper bounds on the price of anarchy. Basically, this framework shows that such bounds can be derived by establishing a certain smoothness condition. An additional strength of this approach is that the smoothness condition allows to derive upper bounds on the price of anarchy for various solution concepts, ranging from pure Nash equilibria to coarse correlated equilibria; the latter being naturally related to outcomes resulting from natural learning algorithms (see, for example, Young [27]). Here, we extend the smoothness framework such that it can be used conveniently in our setting.

Using this extension, we prove upper bounds on the price of anarchy for altruistic versions of three classes of well-studied games: congestion games, minsum scheduling games, and generalized second price auctions. We show that for unrestricted altruism levels the price of anarchy is unbounded. In particular, this happens if there is a player i who cares more about some friend j than about himself, that is, $\alpha_{ij} > \alpha_{ii}$. We therefore derive our upper bounds under the mild assumption that each player cares positively about himself and he cares about any other player at most as much as he cares about himself; we refer to this as *restricted altruistic social context*. Under this assumption, we derive the following upper bounds on the coarse price of anarchy:

- A bound of 7 for altruistic linear congestion games, and a bound of $\varphi^3 \approx 4.236$ for the special case of singleton linear congestion games, where $\varphi = (1+\sqrt{5})/2$ denotes the golden ratio.
- A bound of $4+2\sqrt{3} \approx 7.4641$ and $12+8\sqrt{2} \approx 23.3137$ for altruistic minsum machine scheduling games for related and unrelated machines, respectively.
- A bound of $2(n+1)$ for altruistic generalized second price auctions, where n is the number of players.

Our results therefore show that for congestion games and minsum scheduling games the price of anarchy cannot drastically increase. Specifically, it remains constant, independently of how complex the underlying altruistic social structure is. On the other hand, for generalized second price auctions the price of anarchy may degrade quite drastically: we prove an upper bound of $O(n)$, as opposed to a small constant which is known for the purely selfish setting [6].

Our upper bound proof for singleton congestion games uses a novel proof approach. Typically, for congestion games, smoothness is shown by massaging the smoothness definition such that eventually one can argue for each facility separately. A similar approach turns out to be too weak to derive the bound of φ^3 here. Instead, we use a more refined *amortized* argument by distributing some additional "budget" unevenly among the facilities. We believe that this approach might be of independent interest.

Related Work. There are several papers that propose models of altruism and spite [4,5,7–9,13,16–18]. All these models are special cases of the one studied here. Among these articles, the inefficiency of equilibria in the presence of altruistic/spiteful behavior was studied for various games in [5,7–9,13]. After its introduction in [23], the smoothness framework has been adapted in various directions [24–26], including an extension to a particular model of altruism in [8], which constitutes a special case of the altruistic games considered here.

Biló et al. [2] also studied social context congestion games, in the case where the perceived cost of a player is the minimum, maximum, or sum of the immediate cost of his neighbors. They establish, among other results, an upper bound of 17/3 on the pure price of anarchy of linear congestion games for a special case of the setting we study here.

Related but different from our setting, is the concept of *graphical congestion games* [3,15]. Here the cost and the strategy set of a player depends only on a subset of the players.

After publication of a preliminary conference version of the present paper, follow-up work by Rahn and Schäfer appeared as [22]. In their paper, the authors relate the study of altruistic extensions of games to a class of games named *social contribution games*, and improve (among other things) some of the upper bounds we give here.

2 Preliminaries

Altruistic Extensions of Games. We study the effect of altruistic behavior in strategic games. To model the complex altruistic relationships between the players, we equip the underlying game with an *altruistic social context*. More precisely, let $\Gamma = (N, \{\Sigma_i\}_{i\in N}, \{c_i\}_{i\in N})$ be a strategic game (termed *base game*), where $N = \{1, \dots, n\}$ is the set of players, Σ_i is the strategy set of player i, and $c_i : \Sigma \to \mathbb{R}$ is the direct cost function of player i that maps every strategy profile $s \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ to a real value. Unless stated otherwise, we assume that Γ is a cost minimization game, that is, every player i wants to minimize his individual cost function c_i. Further, we assume that an *altruistic social context* is given by an $n \times n$ matrix $\alpha \in \mathbb{R}^{n\times n}$.

Given a base game Γ and an altruistic social context α, the *α-altruistic extension* of Γ is defined as the strategic game $\Gamma^\alpha = (N, \{\Sigma_i\}_{i\in N}, \{c_i^\alpha\}_{i\in N})$, where for all $i \in N$ and $s \in \Sigma$, the *perceived cost* $c_i^\alpha(s)$ of player i is given by

$$c_i^\alpha(s) = \sum_{j=1}^{n} \alpha_{ij} c_j(s). \tag{1}$$

Thus, the perceived cost of player i in the α-altruistic extension is the α_{ij}-weighted sum of the individual direct costs of all players in the base game. A positive (negative) α_{ij} value signifies that player i cares positively (negatively) about the direct cost of player j, which can be interpreted as an altruistic (spiteful) attitude of i towards j. Note that α_{ii} specifies how player i cares about himself; we also call α_{ii} the *self-perception level*. For simplicity, we will often refer to the resulting game Γ^α as the *α-altruistic game*, without explicitly mentioning the base game Γ and the altruistic social context α.

The above viewpoint has a natural interpretation in terms of *social networks*: Suppose the players in N are identified with the nodes of a complete directed graph $G = (N, A)$. The weight of an edge $(i, j) \in A$ is equal to α_{ij}, specifying the extent to which player i cares about the cost of player j.

The main focus of this paper is on altruistic behavior. We distinguish between *unrestricted* and *restricted* altruistic social contexts α. In the *unrestricted* case we assume that $\alpha_{ij} \geq 0$ for every $i, j \in N$; in particular, the self-perception level of a player can be zero. In this case, one can prove trivial lower bounds for the price of anarchy, just by setting $\alpha_{ij} = 0$, for all i, j. For this reason we consider also the more interesting restricted case. In the *restricted* case, every player has a positive self-perception level and cares about himself at least as much as about any other player, namely, $\alpha_{ii} > 0$ and

$\alpha_{ii} \geq \alpha_{ij} \geq 0$ for every $i, j \in N$, $i \neq j$. In the latter case, we can normalize α without loss of generality such that $\alpha_{ii} = 1$ for every player i.[1]

Coarse Equilibria and the Price of Anarchy. We are interested in the efficiency loss caused by altruistic behavior. Let $C : \Sigma \to \mathbb{R}$ be a *social cost* function that maps strategy profiles to real numbers. Most of the time in this paper, the social cost will refer to the sum of the direct costs of all players, namely, $C(s) = \sum_{i=1}^{n} c_i(s)$. The motivation therefore is that we are interested in the efficiency of the outcome resulting from altruistic behavior, which is modeled through the altered perceived cost functions.

We focus on the inefficiency of *coarse equilibria*, which are defined as follows: Let σ be a probability distribution over Σ. Let σ_{-i} denote the projection of σ onto

$$\Sigma_{-i} = \Sigma_1 \times \cdots \times \Sigma_{i-1} \times \Sigma_{i+1} \times \cdots \times \Sigma_n.$$

Then σ is a *coarse equilibrium* of the altruistic game Γ^α if, for every player i and every strategy $s_i^* \in \Sigma_i$, it holds that

$$\mathbf{E}_{s \sim \sigma}[c_i^\alpha(s)] \leq \mathbf{E}_{s_{-i} \sim \sigma_{-i}}[c_i^\alpha(s_i^*, s_{-i})].$$

We use $CE(\Gamma^\alpha)$ to denote the set of coarse equilibria of Γ^α. Coarse equilibria include several other solution concepts, such as correlated equilibria, mixed Nash equilibria, and pure Nash equilibria.

We study the *price of anarchy* [21] of coarse equilibria of altruistic games. For an altruistic game Γ^α, define

$$POA(\Gamma^\alpha) = \sup_{s \in CE(\Gamma^\alpha)} \frac{C(s)}{C(s^*)},$$

where s^* is a strategy profile that minimizes C. The *coarse price of anarchy* of a class of altruistic games $\mathcal{G}$ is defined as

$$POA(\mathcal{G}) = \sup_{\Gamma^\alpha \in \mathcal{G}} POA(\Gamma^\alpha).$$

Some material has been omitted from the main body of the paper and can be found in the appendix.

3 Smoothness

Smoothness. Roughgarden [23] introduced a general smoothness framework to derive bounds on the coarse price of anarchy. Next we extend this framework to α-altruistic games with arbitrary social cost functions.

[1] To see this, note that, by dividing all α_{ij} by $\alpha_{ii} > 0$, the set of equilibria and the social cost of any outcome remain the same.

Definition 1 Let Γ^α be an α-altruistic extension of a cost minimization game with $\alpha \in \mathbb{R}^{n\times n}$ and social cost function C. Further, let s^* be a strategy profile that minimizes C. Γ^α is (λ, μ)-*smooth* if there exists a strategy profile $\bar{s} \in \Sigma$ such that for every strategy profile $s \in \Sigma$ it holds that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(c_j(\bar{s}_i, s_{-i}) - c_j(s)) \leq \lambda C(s^*) + (\mu - 1)C(s). \tag{2}$$

The following theorem shows that (λ, μ)-smoothness implies a bound on the coarse price of anarchy of α-altruistic games.

Theorem 1 *Let Γ^α be an α-altruistic extension of a cost minimization game with $\alpha \in \mathbb{R}^{n\times n}$ and social cost function C. If Γ^α is (λ, μ)-smooth with $\mu < 1$, then the coarse price of anarchy of Γ^α is at most $\lambda/(1-\mu)$.*

In the purely selfish setting (i.e., when $\alpha_{ii} = 1$ and $\alpha_{ij} = 0$ for every $i, j \in N$, $i \neq j$) our smoothness definition is slightly more general than the one in [23] where (2) is required to hold for any arbitrary strategy profile s^* and with $\bar{s} = s^*$. Also, in [23] the analogue of Theorem 1 is shown under the additional assumption that C is *sum-bounded*, that is, $C(s) \leq \sum_i c_i(s)$. Here, we get rid of this assumption.

Proof (Theorem 1) Let σ be a coarse equilibrium for Γ^α and let s be a random variable with distribution σ. Further, let $\bar{s}$ be a strategy profile for which the smoothness condition (2) holds and let $s^* \in \Sigma$ be an optimal strategy profile. The coarse equilibrium condition implies that for every player $i \in N$:

$$\sum_{j=1}^{n} \alpha_{ij}\mathbf{E}[c_j(\bar{s}_i, s_{-i})] - \sum_{j=1}^{n} \alpha_{ij}\mathbf{E}[c_j(s)] \geq 0.$$

By summing over all players and using linearity of expectation, we obtain

$$\mathbf{E}[C(s)] \leq \mathbf{E}[C(s)] + \mathbf{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(c_j(\bar{s}_i, s_{-i}) - c_j(s))\right].$$

Now we use the smoothness property (2) and obtain

$$\mathbf{E}[C(s)] \leq \mathbf{E}[C(s)] + \mathbf{E}[\lambda C(s^*) + (\mu - 1)C(s)] = \lambda C(s^*) + \mu\mathbf{E}[C(s)].$$

Since $\mu < 1$, this implies that the coarse price of anarchy is at most $\lambda/(1-\mu)$. □

Efficiency of no-regret algorithms. The above smoothness definition allows us to import some additional results from [23]. In particular, it proves useful in the context of natural learning algorithms generating no regret sequences.

Suppose Γ^α satisfies the (λ, μ)-smoothness definition (Definition 1) with $\bar{s} = s^*$. Consider a sequence $s^1, \ldots, s^T$ of outcomes of Γ^α of a game. Let s^* be an optimal outcome that minimizes the social cost function C. Define

$$\delta_i^\alpha(s^t) = c_i^\alpha(s^t) - c_i^\alpha(s_i^*, s_{-i}^t)$$

for every $i \in N$ and $t \in \{1, \ldots, T\}$. Let $\Delta^\alpha(s^t) = \sum_{i=1}^n \delta_i^\alpha(s^t)$. We have

$$\Delta^\alpha(s^t) = \sum_{i=1}^n \left(c_i^\alpha(s^t) - c_i^\alpha(s_i^*, s_{-i}^t)\right) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}(c_j(s^t) - c_j(s_i^*, s_{-i}^t)).$$

Exploiting the (λ, μ)-property, we obtain

$$C(s^t) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{\Delta^\alpha(s^t)}{1-\mu}. \tag{3}$$

Suppose that $s^1, \ldots, s^T$ is a sequence of outcomes in which every player experiences vanishing average external regret, that is, for every player $i \in N$

$$\sum_{t=1}^T c_i^\alpha(s^t) \leq \left[\min_{s_i' \in \Sigma_i} \sum_{t=1}^T c_i^\alpha(s_i', s_{-i}^t)\right] + o(T).$$

We obtain that for every player $i \in N$:

$$\frac{1}{T}\sum_{t=1}^T \delta_i^\alpha(s^t) \leq \frac{1}{T}\left(\sum_{t=1}^T c_i^\alpha(s^t) - \min_{s_i' \in \Sigma_i} \sum_{t=1}^T c_i^\alpha(s_i', s_{-i}^t)\right) = o(1). \tag{4}$$

By summing (3) over all t and using (4), we obtain that the average cost of the sequence of T outcomes is

$$\frac{1}{T}\sum_{t=1}^T C(s^t) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{1}{1-\mu}\sum_{i=1}^n \left(\frac{1}{T}\sum_{t=1}^T \delta_i^\alpha(s^t)\right) \stackrel{T\to\infty}{\longrightarrow} \frac{\lambda}{1-\mu} C(s^*).$$

4 Congestion Games

In a congestion game $\Gamma = (N, E, \{d_e\}_{e \in E}, \{\Sigma_i\}_{i \in N})$ we are given a set of players $N = \{1, \ldots, n\}$, a set of *facilities* E with a *delay function* $d_e : \mathbb{N} \to \mathbb{R}$ for every facility $e \in E$, and a strategy set $\Sigma_i \subseteq 2^E$ for every player $i \in N$. For a strategy profile $s \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$, define $x_e(s)$ as the number of players using facility $e \in E$, that is, $x_e(s) = |\{i \in N : e \in s_i\}|$. The direct cost of player i is defined as $c_i(s) = \sum_{e \in s_i} d_e(x_e(s))$ and the social cost function is given by $C(s) = \sum_{i=1}^n c_i(s)$. In a *linear* congestion game, the delay function of every facility $e \in E$ is of the form $d_e(x) = a_e x + b_e$, where $a_e, b_e \in \mathbb{Q}_{\geq 0}$ are nonnegative rational numbers.

4.1 General Linear Congestion Games

Theorem 2 *Every α-altruistic extension of a linear congestion game with restricted altruistic social context α is $(\frac{7}{3}, \frac{2}{3})$-smooth. Therefore, the coarse price of anarchy is at most 7 for these games.*

We need the following simple technical lemma for the proof of Theorem 2. Its proof is in the appendix.

Lemma 1 *For every two integers $x, y \in \mathbb{N}$*

$$(x+1)y + xy \leq \frac{7}{3}y^2 + \frac{2}{3}x^2. \tag{5}$$

Proof (of Theorem 2) Let s be an arbitrary strategy profile and let s^* be a strategy profile that minimizes C. We can assume without loss of generality that $d_e(x) = x$ for all $e \in E$ [20].

Let x_e and x_e^* refer to $x_e(s)$ and $x_e(s^*)$, respectively. Observe that $c_i(s_i^*, s_{-i}) \leq \sum_{e \in s_i^*}(x_e + 1)$. Taking the sum over all players, we obtain

$$\begin{aligned}\sum_{i=1}^{n}(c_i(s_i^*, s_{-i}) - c_i(s)) &\leq \sum_{i=1}^{n}\sum_{e \in s_i^*}(x_e + 1) - C(s) \\ &= \sum_{e \in E} x_e^*(x_e + 1) - C(s).\end{aligned} \tag{6}$$

Note that the above derivation is a bound on the selfish part of the left hand side of the smoothness condition. Next, we bound the altruistic part of the smoothness condition. Fix some player $i \in N$ and let $x'_e = x_e(s_i^*, s_{-i})$. Note that

$$x'_e = \begin{cases} x_e + 1 & \text{if } e \in s_i^* \setminus s_i\,, \\ x_e - 1 & \text{if } e \in s_i \setminus s_i^*\,, \\ x_e & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}\sum_{j \neq i} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) &= \sum_{j \neq i}\left(\sum_{e \in s_j \cap (s_i^* \setminus s_i)} \alpha_{ij} - \sum_{e \in s_j \cap (s_i \setminus s_i^*)} \alpha_{ij}\right) \\ &= \sum_{e \in s_i^* \setminus s_i}\sum_{j \neq i : e \in s_j} \alpha_{ij} - \sum_{e \in s_i \setminus s_i^*}\sum_{j \neq i : e \in s_j} \alpha_{ij}.\end{aligned}$$

Summing over all players and exploiting that in the restricted case $0 \leq \alpha_{ij} \leq 1$ for every $i, j \in N$, $i \neq j$, we can bound

$$\sum_{i=1}^{n}\left(\sum_{e \in s_i^* \setminus s_i}\sum_{j \neq i : e \in s_j} \alpha_{ij} - \sum_{e \in s_i \setminus s_i^*}\sum_{j \neq i : e \in s_j} \alpha_{ij}\right) \leq \sum_{i=1}^{n}\sum_{e \in s_i^*}\sum_{j : e \in s_j} 1 = \sum_{e \in E} x_e x_e^*. \tag{7}$$

Combining (6) and (7) and using Lemma 1, we conclude that

$$\sum_{i,j} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \leq \frac{7}{3}C(s^*) + \left(\frac{2}{3} - 1\right) C(s)$$

as desired. □

Unrestricted altruistic social context. Theorem 2 considers congestion games with restricted altruistic social context α. This assumption is required; we now present an example showing that the price of anarchy is unbounded if the altruistic social context is unbounded.

Example 1 Let $(N, E, \{d_e\}_{e\in E}, \{\Sigma_i\}_{i\in N}\}$ be a congestion game with $N = \{1, 2\}$ and $E = \{1, 2, 3, 4\}$. Let the delay functions be defined by $d_1(x) = d_3(x) = x$ and $d_2(x) = d_4(x) = Kx$ for all x, where K is a large constant. The strategy sets are $\Sigma_1 = \{\{1, 2\}, \{3\}\}$, $\Sigma_2 = \{\{3, 4\}, \{1\}\}$. Suppose furthermore that α is given by $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = \alpha_{21} = 1$.

Then observe that in the α-altruistic extension of this game, the strategy profile $(\{1, 2\}, \{3, 4\})$ is a pure equilibrium with social cost $2K+2$. The optimal social cost is 2, and is attained by the strategy profile $(\{3\}, \{1\})$. The price of anarchy in this game is therefore $K+1$, and K can be taken arbitrarily large, so the price of anarchy is unbounded.

4.2 Singleton Congestion Games

We derive a better smoothness result for *singleton congestion games with identical delay functions*, that is, when $\Sigma_i \subseteq E$ for every $i \in N$, so that for each strategy $s \in \Sigma_i$ we have that $|s| = 1$.

Theorem 3 *Every α-altruistic extension of a singleton linear congestion game with identical delay functions on all facilities under restricted altruistic social context α is $(1+\varphi, 1/\varphi^2$-smooth, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. Therefore, the coarse price of anarchy is at most $\varphi^3 \approx 4.236$ for these games.*

To prove this theorem, we use a novel proof approach. In most existing proofs one first massages the smoothness condition to derive an equivalent condition summing over all facilities (instead of players), and then establishes smoothness by reasoning for each facility separately. If we follow this approach here, we again obtain an upper bound of 7. Instead, we use an *amortized* argument here to derive our improved bound.

A careful analysis (details can be found in the appendix) can show that the smoothness definition (2) for singleton linear congestion games with $\bar{s} = s^*$ is equivalent to

$$\sum_{i=1}^{n} \sum_{j\neq i} \left(\lambda \left|s_i^* \cap s_j^*\right| + (\mu + \alpha_{ij}) \left|s_i \cap s_j\right| - (1 + \alpha_{ij}) \left|s_i^* \cap s_j\right|\right) + (\lambda+\mu-1)n \geq 0. \tag{8}$$

We translate the proof of this inequality to a coloring problem on a suitably defined graph. We construct an *extended social network* as follows: For every player $i \in N$ we introduce two nodes i and i^* representing player i under s and s^*, respectively. We call the former type of nodes *s-nodes* and the latter type of nodes *s^*-nodes*. For every two players $i, j \in N$ with $i \neq j$ we introduce four edges: (i, j) with weight $2\mu + \alpha_{ij} + \alpha_{ji}$, (i^*, j^*) with weight 2λ, (i^*, j) with weight $-(1 + \alpha_{ij})$, and (i, j^*) with weight $-(1 + \alpha_{ji})$. We identify the set of facilities E with a set of m *colors*, such that $E = [m]$. The colors assigned to i and i^* are s_i and s_i^*, respectively. Call an edge $e = (u, v)$ in the extended network *c-monochromatic* if both u and v have color c. In addition, we distribute a total budget of $(\lambda+\mu-1)n$ among the $2n$ nodes of the extended network.

With the viewpoint of the previous paragraph, the left-hand side of (8) is equal to the total weight of all c-monochromatic edges (summed over all colors c) plus the total budget of all nodes. The idea now is to argue that we can fix λ and μ such that for each color $c \in [m]$ the total weight of all c-monochromatic edges plus the respective node budget is at least 0. The crucial insight to derive our improved bound is that the budget is split unevenly among the nodes: we assign a budget of $(\lambda + \mu - 1)$ to every s-node and 0 to every s^*-node.

Fix some color $c \in [m]$ and consider the subgraph of the extended network induced by the nodes having color c. Partition the nodes into the set S_c of s-nodes and the set S_c^* of s^*-nodes. Imagine we draw this subgraph with all nodes in S_c put on the left-hand side and all nodes in S_c^* put on the right-hand side. The edges from S_c to S_c^* are called *crossing edges*. The edges that stay within S_c or S_c^* are called *internal* edges. Let $x = x_c = |S_c|$ and $y = y_c = |S_c^*|$. Note that the internal edges in S_c constitute a complete graph on x nodes. Similarly, the internal edges in S_c^* constitute a complete graph on y nodes. Note that the crossing edges constitute a $K_{x,y}$ with a few edges missing, namely the pairs (i, i^*) representing the same player i (which are nonexistent by construction). Let $z = z_c$ be the number of such pairs.

In the worst case, $\alpha_{ij} = 0$ for all internal edges and $\alpha_{ij} = 1$ for all crossing edges. The total contribution to the left-hand side of (8) that we can account for color c is then

$$\begin{aligned} 2\mu \cdot \tfrac{1}{2}x(x-1) + 2\lambda \cdot \tfrac{1}{2}y(y-1) - 2 \cdot (xy - z) + (\lambda + \mu - 1) \cdot x \\ = \mu x^2 + \lambda y^2 - 2xy + (\lambda - 1)x - \lambda y + 2z. \end{aligned} \tag{9}$$

We need the following lemma, whose proof can be found in the appendix. It is actually tight, implying that under the smoothness framework we cannot show a better bound. It is a small variation of Lemma 1 in [10].

Lemma 2 *Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. For every two integers $x, y \in \mathbb{N}$, $2xy - \varphi x + \varphi^2 y \leq \frac{1}{\varphi^2}x^2 + \varphi^2 y^2$.*

Fix $\lambda = \varphi^2$ and $\mu = 1/\varphi^2$. Then (9) is nonnegative by Lemma 2. Summing over all colors $c \in [m]$ proves (8). Given our choices of $\lambda = \varphi^2$ and $\mu = 1/\varphi^2$ we obtain a bound on the coarse price of anarchy of $\varphi^3 \approx 4.236$. □

The following example shows that our bound for singleton congestion games is best possible if one sticks to the smoothness framework.

Example 2 Consider a singleton congestion game with n facilities and n players. Set $\alpha_{ij} = 1$ for every $i, j \in N$ with $j = (i+1)_{\bmod n}$ and $\alpha_{ij} = 0$ otherwise. Suppose that player i chooses strategy i in s and strategy $(i+1)_{\bmod n}$ in s^*. For each color class we then have $x_c = y_c = 1$ and $z = 0$ and the bound of Lemma 2 is tight.

5 Minsum Machine Scheduling

In a scheduling game, we deal with a set of machines $[m]$, and a set of jobs $[n]$ that are to be scheduled on the machines. For each job $i \in [n]$ and machine $k \in [m]$, we are given a *processing time* $p_{i,k} \in \mathbb{R}_{\geq 0}$, which is the time it takes to run job i on machine k.

There are many ways in which a machine may execute the set of jobs it gets assigned. We restrict ourselves here to a popular policy where the jobs on a machine are executed one-by-one, in order of increasing processing time (i.e., the longer jobs are executed later). Ties are broken in a deterministic way, and we write $i \prec_k j$ if $p_{i,k} < p_{j,k}$ or $p_{i,k} = p_{j,k}$ and the tie breaking rule schedules job i before job j on machine k. A *schedule* is a vector $s = (s_1, \ldots, s_n)$, where for $i \in [n]$, s_i is the machine on which job i is to be ran. We define the value $N(i, k, s)$ to be the number of jobs j on machine k under strategy profile s for which it holds that $i \prec_k j$. Given s, the *completion time* of a job i under s is

$$p_{i,s_i} + \sum_{j : j \prec_{s_i} i, s_j = s_i} p_{j,s_j}.$$

The jobs take the role of the players: the strategy set of a player is $[m]$, so the strategy profiles that arise are schedules. The cost $c_j(s)$ of a job $j \in [n]$ under strategy profile s is the completion time of j under s.

We define the social cost function for this game to be the sum of the completion times of the jobs. Note that the social cost can be written as

$$C(s) = \sum_{k=1}^{m} \sum_{i : s_i = k} (N(i, k, s) + 1) p_{i,k}.$$

If the processing times are not restricted in any way, we speak of *unrelated machine scheduling games*. We speak of *related machine scheduling games* if the processing times are defined as follows: For each machine $k \in [m]$, there is a *speed* $t_k \in \mathbb{R}_{>0}$ and for each job $j \in [n]$ there is a *length* $p_j \in \mathbb{R}_{\geq 0}$ such that $p_{i,k} = p_j / t_k$ for all $i \in [n]$, $k \in [m]$.

Cole et al. [11] show that unrelated machine scheduling games (without altruism) are $(2, 1/2)$-smooth, resulting in a coarse price of anarchy of at most

4.[2] Hoeksma and Uetz [19] prove that related machine scheduling games (without altruism) are $(2,0)$-smooth, leading to the conclusion that the coarse price of anarchy is at most 2.

Next, we prove constant upper bounds on the price of anarchy for restricted altruistic social contexts.

Theorem 4 *Every α-altruistic extension of a machine scheduling game with restricted altruistic social context α is $(2+x, 1/x)$-smooth for related machines and $(2+x, 1/2+1/x)$-smooth for unrelated machines for every $x \in \mathbb{R}_{>0}$. Therefore, the coarse price of anarchy is at most $4+2\sqrt{3} \approx 7.4641$ (choosing $x = 1+\sqrt{3}$) and $12+8\sqrt{2} \approx 23.3137$ (choosing $x = 2+2\sqrt{2}$) for these games, respectively.*

Proof Recall that we assume that α is a restricted social context such that $\alpha_{ii} = 1$ for all $i \in [n]$ and $\alpha_{ij} \in [0,1]$ for all $i,j \in [n]$.

In [19] it is proved that the base game for the case of related machines is $(2,0)$-smooth, and from the proof of Theorem 3.2 in [11], it follows that the base game for the case of unrelated machines is $(2,1/2)$-smooth. Thus, let s^* be an arbitrary optimal schedule and let s be an arbitrary schedule. It suffices to show that

$$\sum_{i=1}^{n} \sum_{j\in[n]:j\neq i} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \leq xC(s^*) + \frac{C(s)}{x}$$

for all $x > 0$.

Let

$$\begin{aligned} P_1 &= \{(i,j) : s_i^* = s_j, s_i^* \neq s_i, i \prec_{s_i^*} j\}, \quad \text{and} \\ P_2 &= \{(i,j) : s_i = s_j, s_i^* \neq s_j, i \prec_{s_i} j\}. \end{aligned}$$

Informally, P_1 (P_2) is the set of pairs of jobs (i,j) such that i's strategy change from s_i to s_i^* makes j become scheduled later (earlier). Note that for a pair of players (i,j) that is not in $P_1 \cup P_2$, we have $c_j(s_i^*, s_{-i}) - c_j(s) = 0$, and for $(i,j) \in P_2$ it holds that $\alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \leq 0$.

Therefore:

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j\in[n]:j\neq i} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) &= \sum_{(i,j)\in P_1\cup P_2} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \\ &\leq \sum_{(i,j)\in P_1} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \leq \sum_{(i,j)\in P_1} (c_j(s_i^*, s_{-i}) - c_j(s)) \\ &= \sum_{(i,j)\in P_1} p_{i,s_i^*}. \end{aligned}$$

[2] More precisely, this is shown to hold for the more general case when the social cost is an arbitrary nonnegative linear combination of the player's cost. From a scheduling game instance described in [12], it follows that this bound is tight, i.e., that the coarse price of anarchy is actually exactly 4.

We now rewrite this last expression into a summation over the machines. We obtain:

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j\in[n]:j\neq i} \alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \\
&\quad \leq \sum_{k=1}^{m} \sum_{(i,j)\in P_1: s_i^*=k} p_{i,k} \\
&\quad = \sum_{k=1}^{m} \sum_{i\in[n]:s_i^*=k} \sum_{j\in[n]:(i,j)\in P_1} p_{i,k} \\
&\quad = \sum_{k=1}^{m} \sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k}} \sum_{j:s_j=k, i\prec_k j} p_{i,k} \\
&\quad \leq \sum_{k=1}^{m} \sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k}} N(i,k,s)p_{i,k} \\
&\quad = \sum_{k=1}^{m} \sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k}} (xN(i,k,s^*) + x - 1 + N(i,k,s) - xN(i,k,s^*) - x + 1)p_{i,k} \\
&\quad \leq \sum_{k=1}^{m} \sum_{i\in[n]:s_i^*=k} (x(N(i,k,s^*)+1) - 1)p_{i,k} \\
&\qquad + \sum_{k=1}^{m} \sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k}} (N(i,k,s) - xN(i,k,s^*) - x + 1)p_{i,k} \\
&\quad \leq \sum_{k=1}^{m} \sum_{i\in[n]:s_i^*=k} (x(N(i,k,s^*)+1) - 1)p_{i,k} \\
&\qquad + \sum_{k=1}^{m} \sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ N(i,k,s)>xN(i,k,s^*)+x-1}} \lceil N(i,k,s) - xN(i,k,s^*) - x + 1\rceil p_{i,k}.
\end{aligned}$$

Consider a job i and machine k such that it holds that $s_i^* = k$, $s_i \neq k$, and $N(i,k,s) > xN(i,k,s^*) + x - 1$. Let $S(i,k)$ be the set of $\lceil N(i,k,s) - xN(i,k,s^*) - x\rceil$ smallest jobs $j \succ_k i$ such that $s_j = k$. Note that $S(i,k)$ is well defined in the sense that this number of jobs exists because $N(i,k,s) > xN(i,k,s^*) + x - 1$ implies $\lceil N(i,k,s) - xN(i,k,s^*) - x\rceil \geq 0$, and because there exist $N(i,k,s) \geq |S(i,k)|$ jobs $j \succ_k i$ with $s_j = k$. The the last expression equals

$$\sum_{k=1}^{m} \sum_{i\in[n]:s_i^*=k} (x(N(i,k,s^*)+1) - 1)p_{i,k}$$

$$+\sum_{k=1}^{m}\sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ N(i,k,s)>xN(i,k,s^*)+x-1}}\left(p_{i,k}+\sum_{j\in S(i,k)}p_{i,k}\right),$$

which is less than or equal to

$$xC(s^*)+\sum_{k=1}^{m}\sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ N(i,k,s)>xN(i,k,s^*)+x-1}}\sum_{j\in S(i,k)}p_{i,k},$$

thus, obtaining,

$$\sum_{i=1}^{n}\sum_{j\in[n]:j\neq i}\alpha_{ij}(c_j(s_i^*,s_{-i})-c_j(s))$$
$$\leq xC(s^*)+\sum_{k=1}^{m}\sum_{j\in S(i,k)}\sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ N(i,k,s)>xN(i,k,s^*)+x-1}}p_{i,k}. \tag{10}$$

Note that for every job $j\in S(i,k)$ it holds that $N(j,k,s)\geq N(i,k,s)-|S(i,k)|>xN(i,k,s^*)+x-1$. Therefore, the expression on the right-hand side of (10) is equivalent to

$$xC(s^*)+\sum_{k=1}^{m}\sum_{j\in S(i,k)}\sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ N(i,k,s)>xN(i,k,s^*)+x-1\\ N(j,k,s)>xN(i,k,s^*)+x-1}}p_{i,k}. \tag{11}$$

Note that $j\in S(i,k)$ implies that $s_j=k$. We relax some of the constraints in the summations and so we upper bound the expression in (11) by

$$xC(s^*)+\sum_{k=1}^{m}\sum_{j\in[n]:s_j=k}\sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ i\prec_k j,\\ N(j,k,s)>xN(i,k,s^*)+x-1}}p_{i,k}$$
$$\leq xC(s^*)+\sum_{k=1}^{m}\sum_{j\in[n]:s_j=k}\sum_{\substack{i\in[n]:s_i^*=k,\\ s_i\neq k,\\ i\prec_k j,\\ N(j,k,s)>xN(i,k,s^*)+x-1}}p_{j,k}. \tag{12}$$

The next step in the derivation is made by observing that for each job j and each machine k such that $s_j=k$, there are at most $\lceil(N(j,k,s)-x+1)/x\rceil$ jobs $i\prec_k j$ such that $s_i^*=k$, $s_i\neq k$ and $N(j,k,s)>xN(i,k,s^*)+x-1$. To see this, assume for contradiction that there are *more* than $\lceil(N(j,k,s)-x+1)/x\rceil$

jobs $i \prec_k j$ such that $s_i^* = k$, $s_i \neq k$ and $N(j,k,s) > xN(i,k,s^*) + x - 1$. Let i be the $(\lceil (N(j,k,s) - x + 1)/x \rceil + 1)$-th largest job for which these three properties hold. Then, there are at least $\lceil (N(j,k,s) - x + 1)/x \rceil$ jobs scheduled on machine k that have these properties and that are scheduled after i on machine k under strategy s^*. Therefore, we have that $xN(i,k,s^*) + x - 1 \geq x\lceil (N(j,k,s) - x + 1)/x \rceil + x - 1 \geq N(j,k,s)$, which is a contradiction.

Using this observation together with (10), (11) and (12), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} \sum_{j \in [n]: j \neq i} \alpha_{ij} (c_j(s_i^*, s_{-i}) - c_j(s)) \\
&\quad \leq xC(s^*) + \sum_{k=1}^{m} \sum_{j \in [n]: s_j = k} \left\lceil \frac{N(j,k,s) - x + 1}{x} \right\rceil p_{j,k} \\
&\quad \leq xC(s^*) + \sum_{k=1}^{m} \sum_{j \in [n]: s_j = k} \frac{1}{x} (N(j,k,s) + 1) p_{j,k} \\
&\quad = xC(s^*) + \frac{C(s)}{x},
\end{aligned}
$$

where for the second inequality we use the basic fact that $\lceil a \rceil \leq a + 1$ for all $a \in \mathbb{R}$. □

Similarly to congestion games, the assumption of the altruistic social context being restricted is necessary. Next we provide an example showing that the price of anarchy is unbounded if the altruistic social context is unrestricted.

Example 3 Fix a number $M \in \mathbb{R}_{>0}$ arbitrarily, and consider the scheduling game with two machines and two jobs. The speeds are given by $t_1 = M, t_2 = 1$. The job lengths are given by $p_1 = p_2 = 1$. Suppose that the altruism levels are as follows: $\alpha_{11} = \alpha_{22} = \alpha_{21} = 0$, $\alpha_{12} = 1$. Then the schedule where job 1 is on machine 1 and job 2 is on machine 2 is a pure equilibrium. The social cost of this equilibrium is $M + 1$. When $M \geq 2$, it is a social optimum to schedule both jobs on machine 2, and this schedule achieves a social cost of 3. Therefore, for $M \geq 2$, the price of anarchy of this altruistic scheduling game is $M + 1/3$. Because M can be picked arbitrarily large, this shows that the price of anarchy is arbitrarily bad for altruistic extensions of scheduling games, when we allow arbitrary altruism levels.

6 Profit-Maximization Games and Generalized Second-Price Auctions

In this section we study generalized second-price auctions. This is a profit-maximization game. We will, therefore, first extend our notions to profit-maximization games. Then we will apply the extended framework to second-price auctions.

6.1 Profit-Maximization Games

The smoothness definition introduced in Section 2 can naturally be adapted to profit maximization games.

Let $\Gamma = (N, \{\Sigma_i\}_{i \in N}, \{p_i\}_{i \in N})$ be a strategic game, where each player i seeks to maximize his direct profit function $p_i : \Sigma \to \mathbb{R}$. The α-altruistic extension Γ^α of Γ is defined similarly as for cost minimization games.

Definition 2 Let Γ^α be an α-altruistic extension of a profit maximization game with $\alpha \in \mathbb{R}^{n \times n}$ and social welfare function Π. Further, let s^* be a strategy profile that maximizes Π. Γ^α is (λ, μ)-*smooth* if there exists a strategy profile $\bar{s} \in \Sigma$ such that for every strategy profile $s \in \Sigma$ it holds that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} (p_j(\bar{s}_i, s_{-i}) - p_j(s)) \geq \lambda \Pi(s^*) - (\mu + 1) \Pi(s). \tag{13}$$

The proof of the following theorem proceeds along the same lines as the one of Theorem 1 and is omitted.

Theorem 5 *Let Γ^α be an α-altruistic extension of a profit maximization game with $\alpha \in \mathbb{R}^{n \times n}$ and social welfare function π. If Γ^α is (λ, μ)-smooth with $\mu > -1$, then the coarse price of anarchy of Γ^α is at most $(1 + \mu)/\lambda$.*

We are now ready to study generalized second-price auctions.

6.2 Generalized Second-Price Auctions

We study auctions where a set $N = [n]$ of n bidders compete for k slots. Each bidder $i \in N$ has a valuation $v_i \in \mathbb{R}_{\geq 0}$ and specifies a bid $b_i \in \mathbb{R}_{\geq 0}$. Each slot $j \in [k]$ has a *click-through rate* $\gamma_j \in \mathbb{R}_{\geq 0}$. Without loss of generality, we assume that the slots are sorted according to their click-through rates such that $\gamma_1 \geq \cdots \geq \gamma_k$ and that $k = n$.[3]

We consider the *generalized second price auction (GSP)* as the underlying mechanism. Given a bidding profile $b = (b_1, \ldots, b_n)$, GSP orders the bidders by nonincreasing bids and assigns them in this order to the slots. Each bidder pays the next highest bid for his slot. More precisely, let $b_1 \geq \cdots \geq b_n$ be the ordered list of bids. We assume without loss of generality that if $b_i = b_j$ for two bidders $i > j$ then i precedes j in the order. Then bidder i is assigned to slot i and has to pay b_{i+1}, where we define $b_{n+1} = 0$. The utility of player i for bidding profile b is defined as $u_i(b) = \gamma_i(v_i - b_{i+1})$. The *social welfare* for a bidding profile b is defined as $\Pi(b) = \sum_{i=1}^{n} \gamma_i v_i$.

A standard assumption in this setting is that bidders do not *overbid* their valuations, that is, the strategy set of bidder i is $[0, v_i]$ for all $i \in [n]$. This assumption is made for reasons related to individual rationality: overbidding is

[3] If $k < n$ we can add $n - k$ dummy slots with click-through rate 0; if $k > n$ we can remove the $k - n$ last slots.

a dominated strategy. The same reasoning still applies if players are altruistic. Therefore, we will assume that bidders do not overbid.

We now prove an upper bound of $O(n)$ on the coarse price of anarchy of α-altruistic extensions of generalized second price auctions if the altruistic social context is restricted.

Theorem 6 *Every α-altruistic extension of a generalized second price auction with restricted altruistic social context α is $(\frac{1}{2}, n)$-smooth. Therefore, the coarse price of anarchy is at most $2n + 1$ for these games.*

Proof Let b^* and b be two bidding profiles. By renaming, we assume that for all j, bidder j gets assigned to slot j under bidding profile b.

The base game is known to be $(\lambda_1, \mu_1) = (\frac{1}{2}, 1)$-smooth [24]. That is, for every two bidding profiles b, b^*, it holds that $\sum_{i \in N} u_i(b_i^*, b_{-i}) \geq \frac{1}{2}\Pi(b^*) - \Pi(b)$.

It remains to bound

$$\sum_{i=1}^{n}\sum_{j \neq i} \alpha_{ij}(u_j(b_i^*, b_{-i}) - u_j(b)) \geq \sum_{i=1}^{n}\sum_{j \neq i} \alpha_{ij}(-u_j(b)) \geq \sum_{i=1}^{n}\sum_{j \neq i} \alpha_{ij}(-\gamma_j v_j)$$

$$\geq \sum_{i=1}^{n}\sum_{j \neq i} -\gamma_j v_j \geq -(n-1)\Pi(b).$$

Combining these inequalities proves $(\lambda, \mu) = (\frac{1}{2}, n)$-smoothness and applying Theorem 5 gives the result. □

As in the case of congestion games, the analysis is essentially tight, as we show with the next example.

Example 4 Let n be even and define $k = n/2$. We define an instance of $n + 1$ bidders as follows. Let

$$\gamma_1 = \cdots = \gamma_k = 1, \quad \gamma_{k+1} = \cdots = \gamma_n = \varepsilon \quad \text{and} \quad \gamma_{n+1} = \varepsilon^2.$$

Subsequently, we will make sure that ε, $0 < \varepsilon < 1$, is chosen sufficiently small. Further, define the bidder valuations as

$$v_1 = \cdots = v_{k-1} = \varepsilon, \quad v_k = \cdots = v_n = 1 \quad \text{and} \quad v_{n+1} = 0.$$

We assume that no bidder overbids his valuation. In particular, this implies that bidder $n + 1$ bids zero always.

Let $b^* = (b_1^*, \ldots, b_{n+1}^*)$ be an optimal bidding profile maximizing social welfare. Note that $b_i^* > b_{n+1}^* = 0$ for every $i \in \{1, \ldots, n\}$ because otherwise bidder $n + 1$ would be assigned to a slot with click-through rate at least ε, which is a contradiction to the optimality of b^*. The total social welfare of b^* is $\Pi(b^*) = k + \varepsilon + (k-1)\varepsilon^2$.

Fix a bidding profile $b = (b_1, \ldots, b_{n+1})$ such that $b_1 > \cdots > b_{n+1} = 0$ and $b_1 < b_i^*$ for every $i \in \{1, \ldots, n\}$. Note that this is always possible because $b_i^* > 0$ for every $i \in \{1, \ldots, n\}$. Also note that this also implies that $b_i < \varepsilon$ for all i because we assume that bidders do not overbid. Thus, by choosing ε

sufficiently small, we ensure that the bids in b become arbitrarily small but induce the order as indicated above. We have

$$\Pi(b) = \sum_{i=1}^{n+1} \gamma_i v_i = (n-1)\varepsilon + 1.$$

If bidder $i \in \{1, \dots, n\}$ changes his bid from b_i to b_i^*, then he is assigned to slot 1 in (b_i^*, b_{-i}), thereby shifting the players $1, \dots, i-1$ one slot down (relative to the slots they are assigned to under b). Observe that bidder k is assigned to the last slot with click-through rate 1. As a consequence, whenever one of the bidders $i \in \{k+1, \dots, n\}$ deviates to (b_i^*, b_{-i}), bidder k is shifted from his slot k with click-through rate 1 down to slot $k+1$ with click-through rate ε. We will exploit this below.

Consider the left-hand side of the smoothness definition

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \alpha_{ij}(u_j(b_i^*, b_{-i}) - u_j(b)).$$

Fix player i and consider the contribution $\Delta_j(i) = \alpha_{ij}(u_j(b_i^*, b_{-i}) - u_j(b))$ of player j to the sum. Clearly, $\Delta_j(n+1) = 0$ for every j because bidder $n+1$ is assigned to slot $n+1$ under both profiles. Let $i \in \{1, \dots, n\}$. We distinguish four cases depending on the position of j with respect to i:

Case $j > i$: The deviation of i does not affect player j and thus $\Delta_j(i) = 0$.

Case $j = i$: Player i moves up to the first slot for which he pays b_1. Thus

$$\Delta_i(i) = \alpha_{ii}(\gamma_1(v_i - b_1) - \gamma_i(v_i - b_{i+1})).$$

Case $j = i-1$: Player j moves down one slot for which he pays b_{j+2} (instead of b_{j+1} under b). Thus

$$\Delta_j(i) = \alpha_{ij}(\gamma_{j+1}(v_j - b_{j+2}) - \gamma_j(v_j - b_{j+1})).$$

Case $j < i-1$: Player j moves down one slot for which he pays b_{j+1} (as under b). Thus

$$\Delta_j(i) = \alpha_{ij}(\gamma_{j+1}(v_j - b_{j+1}) - \gamma_j(v_j - b_{j+1})) = \alpha_{ij}(\gamma_{j+1} - \gamma_j)(v_j - b_{j+1}).$$

Choosing ε sufficiently small we can make sure that the total contribution of the payments b_ℓ in each $\Delta_j(i)$ in each of the above four cases is negligible.

We consider the restricted social context, so without loss of generality we assume that all α_{ii} are normalized to 1. Recall that $\gamma_1 = 1$. Ignoring the effect of the payments (which we just argued is negligible if we make ε small enough), the total contribution to the left-hand side of the smoothness definition is thus

$$\sum_{i=1}^{n} \left(\alpha_{ii}(\gamma_1 v_i - \gamma_i v_i) + \sum_{j=1}^{i-1} \alpha_{ij}(\gamma_{j+1} - \gamma_j) v_j \right)$$
$$= \sum_{i=1}^{n} (1 - \gamma_i) v_i + \sum_{i=1}^{n} \sum_{j=1}^{i-1} \alpha_{ij}(\gamma_{j+1} - \gamma_j) v_j$$

Note that $\gamma_{j+1} - \gamma_j = 0$ for all $j \neq k$ and $\gamma_{k+1} - \gamma_k = \varepsilon - 1$. The above expression thus simplifies to

$$\sum_{i=k+1}^{n} (1-\varepsilon)v_i + \sum_{i=k+1}^{n} \alpha_{ik}(\gamma_{k+1} - \gamma_k)v_k = k(1-\varepsilon) - (1-\varepsilon)\sum_{i=k+1}^{n} \alpha_{ik}$$
$$= (1-\varepsilon)\left(k - \sum_{i=k+1}^{n} \alpha_{ik}\right).$$

By setting $\alpha_{ik} = 1$ for every i, the above contribution is equal to zero.

To show (λ, μ)-smoothness, we need to lower bound the latter expression by $\lambda \Pi(b^*) - (\mu+1)\Pi(b)$. That is, λ and μ need to satisfy

$$(1-\varepsilon)\left(k - \sum_{i=k+1}^{n} \alpha_{ik}\right) = 0 \geq \lambda(k + \varepsilon + (k-1)\varepsilon^2) - (\mu+1)((n-1)\varepsilon + 1).$$

which implies (letting $\varepsilon \to 0$) that $\mu + 1 \geq \lambda k$. This provides an asymptotic lower bound of

$$\frac{1+\mu}{\lambda} \geq \frac{k}{1} = \frac{n}{2}$$

on the possible price of anarchy achievable by our smoothness framework.

7 Conclusions

We studied the coarse price of anarchy of altruistic extensions of three fundamental classes of games. The main focus of this paper was put on deriving upper bounds that are *independent* of the underlying social network structure. An interesting open question is whether one can derive refined bounds by exploiting *structural properties* of the underlying social network.

In the present studies, we concentrated on altruistic games with nonnegative altruistic social contexts α, even though our model of altruistic games and the smoothness definition introduced in Sections 2 and 3 allow us to incorporate spiteful behavior as well. We leave it as an interesting open direction for future research to pursue such analyses for spiteful behavior.

References

1. Ashlagi, I., Krysta, P., Tennenholtz, M.: Social context games. In: Proceedings of the 4th International Workshop on Internet and Network Economics. pp. 675–683 (2008)
2. Bilò, V., Celi, A., Flammini, M., Gallotti, V.: Social context congestion games. In: Proceedings of the 18th international conference on Structural Information and Communication Complexity. pp. 282–293. Springer (2011)
3. Bilò, V., Fanelli, A., Flammini, M., Moscardelli, L.: Graphical congestion games. Algorithmica 61(2), 274–297 (2011)
4. Brandt, F., Sandholm, T., Shoham, Y.: Spiteful bidding in sealed-bid auctions. In: Proceedings of the 20th International Joint Conference on Artifical Intelligence. pp. 1207–1214 (2007)

5. Buehler, R., Goldman, Z., Liben-Nowell, D., Pei, Y., Quadri, J., Sharp, A., Taggart, S., Wexler, T., Woods, K.: The price of civil society. In: Proceedings of the 7th International Workshop on Internet and Network Economics. pp. 375–382 (2011)
6. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P., Kyropoulou, M.: On the efficiency of equilibria in generalized second price auctions. In: Proceedings of the 12th ACM Conference on Electronic Commerce. pp. 81–90 (2011)
7. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P., Kyropoulou, M., Papaioannou, E.: The impact of altruism on the efficiency of atomic congestion games. In: Proceedings of the 5th Symposium on Trustworthy Global Computing (2010)
8. Chen, P.A., de Keijzer, B., Kempe, D., Schäfer, G.: The robust price of anarchy of altruistic games. In: Proceedings of the 7th International Workshop on Internet and Network Economics. pp. 383–390 (2011)
9. Chen, P.A., Kempe, D.: Altruism, selfishness, and spite in traffic routing. In: Proceedings of the 9th ACM Conference on Electronic Commerce. pp. 140–149 (2008)
10. Christodoulou, G., Mirrokni, V.S., Sidiropoulos, A.: Convergence and approximation in potential games. Theoretical Computer Science 438(0), 13 –27 (2012)
11. Cole, R., Correa, J.R., Gkatzelis, V., Mirrokni, V., Olver, N.: Inner product spaces for minsum coordination mechanisms. In: Proceedings of the 43rd ACM Symposium on the Theory of Computing. pp. 539–548 (2011)
12. Correa, J.R., Queyranne, M.: Efficiency of equilibria in restricted uniform machine scheduling with total weighted completion time as social cost. Naval Research Logistics 59(5), 384–395 (2012)
13. Elias, J., Martignon, F., Avrachenkov, K., Neglia, G.: Socially-aware network design games. In: Proceedings of the 29th Conference on Computer Communications. pp. 41–45 (2010)
14. Fehr, E., Schmidt, K.M.: The Economics of Fairness, Reciprocity and Altruism: Experimental Evidence and New Theories, Handbook on the Economics of Giving, Reciprocity and Altruism, vol. 1, chap. 8, pp. 615–691. Elsevier (2006)
15. Fotakis, D., Gkatzelis, V., Kaporis, A.C., Spirakis, P.: The impact of social ignorance on weighted congestion games. In: Proceedings of the 5th International Workshop on Internet and Network Economics. pp. 316–327. Springer (2009)
16. Hoefer, M., Skopalik, A.: Altruism in atomic congestion games. In: Proceedings of the 17th European Symposium on Algorithms. pp. 179–189 (2009)
17. Hoefer, M., Skopalik, A.: Stability and convergence in selfish scheduling with altruistic agents. In: Proceedings of the 5th International Workshop on Internet and Network Economics. pp. 616–622 (2009)
18. Hoefer, M., Skopalik, A.: Social context in potential games. In: Proceedings of the 8th International Conference on Internet and Network Economics. pp. 364–377 (2012)
19. Hoeksma, R., Uetz, M.: The price of anarchy for minsum related machine scheduling. In: Proceedings of the 9th International Conference on Approximation and Online Algorithms. pp. 261–273 (2012)
20. de Keijzer, B.: Externalities and Cooperation in Algorithmic Game Theory. Ph.D. thesis, Centrum voor Wiskunde en Informatica (CWI) (2014)
21. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: Proceedings of the 16th Symposium on Theoretical Aspects of Computer Science. pp. 404–413 (1999)
22. Rahn, M., Schäfer, G.: Bounding the inefficiency of altruism through social contribution games. In: Chen, Y., Immorlica, N. (eds.) Proceedings of the 9th International Conference on Web and Internet Economics (WINE), Lecture Notes in Computer Science, vol. 8289, pp. 391–404. Springer (2013)
23. Roughgarden, T.: Intrinsic robustness of the price of anarchy. In: Proceedings of the 41st ACM Symposium on the Theory of Computing. pp. 513–522 (2009)
24. Roughgarden, T.: The price of anarchy in games of incomplete information. In: Proceedings of the 13th ACM Conference on Electronic Commerce. pp. 862–879 (2012)
25. Syrgkanis, V.: Bayesian games and the smoothness framework. CoRR abs/1203.5155 (2012)
26. Syrgkanis, V., Tardos, É.: Composable and efficient mechanisms. In: Proceedings of the 45th ACM Symposium on the Theory of Computing (2013)
27. Young, H.P.: Strategic Learning and its Limits. Oxford University Press (1995)

A Technical Lemmas and Calculations

Here we accumulate all technical details that were omitted from the discussion above.

Proof of Lemma 1 We write x as $(y+z)$, where $z \in \mathbb{Z}$. Substituting this into (5) results in

$$((y+z)+1)y + (y+z)y \leq \frac{7}{3}y^2 + \frac{2}{3}(y+z)^2, \tag{14}$$

which can be rewritten as

$$\frac{2}{3}yz + y - y^2 \leq \frac{2}{3}z^2. \tag{15}$$

When $z \leq 0$, (15) is easily seen to hold, because the first term of the left hand side is nonpositive, the second and third terms together are nonpositive (taking into account that y is nonnegative), and the right hand side is nonnegative.

When $z > 0$ and $y \leq z$, (15) holds because

$$\frac{2}{3}yz + y - y^2 \leq \frac{2}{3}yz \leq \frac{2}{3}z^2.$$

Lastly, when $z > 0$ and $y > z$, (15) holds because

$$\frac{2}{3}yz + y - y^2 < \frac{2}{3}y^2 + y - y^2 = y - \frac{1}{3}y^2.$$

When $y \geq 3$, the latter expression is nonpositive and thus not exceeds $\frac{2}{3}z^2$. For the values of (y,z) not yet covered in this case analysis (i.e., $(y,z) \in \{(1,1),(2,1)\}$, it can be checked by hand that (15) holds. □

Deriving Inequality (8). Note that to satisfy the smoothness condition (2) it suffices to show that

$$\sum_{i=1}^{n}\left(c_i(s_i^*, s_{-i}) + \sum_{j\neq i}\alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s))\right) \leq \lambda C(s^*) + \mu C(s).$$

Let s and s^* be two strategy profiles and let x_e and x_e^* refer to $x_e(s)$ and $x_e(s^*)$, respectively. Fix some player $i \in N$ and let $x_e' = x_e(s_i^*, s_{-i})$. Note that

$$x_e' = \begin{cases} x_e + 1 & \text{if } e \in s_i^* \setminus s_i \\ x_e - 1 & \text{if } e \in s_i \setminus s_i^* \\ x_e & \text{otherwise.} \end{cases}$$

Using these relations, we obtain

$$\begin{aligned}
&c_i(s_i^*, s_{-i}) + \sum_{j\neq i}\alpha_{ij}(c_j(s_i^*, s_{-i}) - c_j(s)) \\
&\quad= \sum_{e\in s_i^*} x_e' + \sum_{j\neq i}\alpha_{ij}\sum_{e\in s_j}(x_e' - x_e) \\
&\quad= \sum_{e\in s_i^*\setminus s_i}(x_e+1) + \sum_{e\in s_i^*\cap s_i} x_e + \sum_{j\neq i}\alpha_{ij}\left(\sum_{e\in s_i^*\cap s_j} 1 - \sum_{e\in s_i\cap s_j} 1\right) \\
&\quad= \sum_{e\in s_i^*\setminus s_i}\left(1 + \sum_{j\neq i: e\in s_j} 1\right) + \sum_{e\in s_i^*\cap s_i}\left(1 + \sum_{j\neq i: e\in s_j} 1\right) + \sum_{j\neq i}\alpha_{ij}\left(\sum_{e\in s_i^*\cap s_j} 1 - \sum_{e\in s_i\cap s_j} 1\right) \\
&\quad= \sum_{e\in s_i^*}\left(1 + \sum_{j\neq i: e\in s_j} 1\right) + \sum_{j\neq i}\alpha_{ij}\left(\sum_{e\in s_i^*\cap s_j} 1 - \sum_{e\in s_i\cap s_j} 1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{e\in s_i^*} 1 + \sum_{j\neq i}\left(\sum_{e\in s_i^*\cap s_j}(1+\alpha_{ij}) - \sum_{e\in s_i\cap s_j}\alpha_{ij}\right)\\
&= |s_i^*| + \sum_{j\neq i}\left((1+\alpha_{ij})\,|s_i^*\cap s_j| - \alpha_{ij}\,|s_i\cap s_j|\right).
\end{aligned}
$$

Note that

$$
C(s) = \sum_{i=1}^n \sum_{e\in s_i}\sum_{j:e\in s_j} 1 = \sum_{i=1}^n\left(\sum_{e\in s_i} 1 + \sum_{j\neq i}\sum_{e\in s_i\cap s_j} 1\right) = \sum_{i=1}^n\sum_{j\neq i}|s_i\cap s_j| + \sum_{i=1}^n |s_i|.
$$

$C(s^*)$ can be expressed similarly.

That is, our smoothness definition is equivalent to

$$
\begin{aligned}
&\sum_{i=1}^n\sum_{j\neq i}\left((1+\alpha_{ij})\,|s_i^*\cap s_j| - \alpha_{ij}\,|s_i\cap s_j|\right) + \sum_{i=1}^n |s_i^*|\\
&\qquad \leq \lambda\left(\sum_{i=1}^n\sum_{j\neq i}\left|s_i^*\cap s_j^*\right| + \sum_{i=1}^n |s_i^*|\right) + \mu\left(\sum_{i=1}^n\sum_{j\neq i}|s_i\cap s_j| + \sum_{i=1}^n |s_i|\right).
\end{aligned}
$$

Rearranging terms yields

$$
\begin{aligned}
&\sum_{i=1}^n\sum_{j\neq i}\left(\lambda\left|s_i^*\cap s_j^*\right| + (\mu+\alpha_{ij})\,|s_i\cap s_j| - (1+\alpha_{ij})\,|s_i^*\cap s_j|\right)\\
&\qquad + (\lambda-1)\sum_{i=1}^n |s_i^*| + \mu\sum_{i=1}^n |s_i| \geq 0.
\end{aligned}
$$

For singleton congestion games, this is equivalent to (8).

Proof of Lemma 2. It is easy to check that the claim holds if $x=0$ or $y=0$. Let $x\geq 1$ and $y\geq 1$. Recall that $1+\varphi=\varphi^2$. We have

$$
\begin{aligned}
\varphi^2y^2 - 2xy + \frac{1}{\varphi^2}x^2 + \varphi x - \varphi^2 y &= \left(\varphi y - \frac{1}{\varphi}x\right)^2 + \varphi x - (1+\varphi)y\\
&\geq 2\varphi y - \frac{2}{\varphi}x - 1 + \varphi x - (1+\varphi)y = (\varphi-1)y + \left(\varphi - \frac{2}{\varphi}\right)x - 1\\
&= \frac{1}{\varphi}y + \left(1-\frac{1}{\varphi}\right)x - 1 \geq 0,
\end{aligned}
$$

where the first inequality follows because $z^2\geq 2z-1$ for every $z\in\mathbb{R}$ and the last one holds because $x\geq 1$ and $y\geq 1$. □